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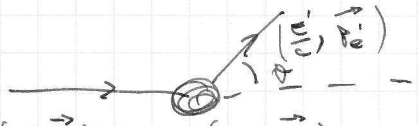
Recoil Energy ($e^- p$ scattering)



$p_e = \left(\frac{E}{c}, \vec{p}_e\right) \qquad \left(\frac{E_p}{c}, \vec{p}_p\right) \longrightarrow \left(\frac{E_p'}{c}, \vec{p}_p'\right)$

$(Mc, 0)$

Using: $E \gg m_e c^2 \Rightarrow E \simeq |\vec{p}| \cdot c \qquad (E \simeq \text{MeV})$

$$(p_e + p_p)^2 = (p_e' + p_p')^2 \quad \longleftarrow \text{4 momenta !}$$

$$p_e^2 + 2 p_e p_p + p_p^2 = p_e'^2 + 2 p_e' p_p' + p_p'^2$$

Ellastic scattering

$p_e^2 = p_e'^2 = m_e^2 c^2$

$p_p^2 = p_p'^2 = M^2 c^2$

$m_e = \frac{\sqrt{p^2}}{c} = \frac{\sqrt{\frac{E^2}{c^2} - \vec{p}^2}}{c}$ invariant mass

$$\Rightarrow \quad p_e p_p = p_e' p_p'$$

$p_p' = p_e + p_p - p_e'$ momentum is conserved

$$p_e \cdot p_p = p_e' \cdot (p_e + p_p - p_e') = p_e' p_e + p_e' p_p - m_e^2 c^2$$

$p_e = \left(\frac{E}{c}, \vec{p}_e\right) \qquad p_p = (Mc, 0)$

$p_e' = \left(\frac{E'}{c}, \vec{p}_e'\right) \qquad p_p' = \left(\frac{E_p'}{c}, \vec{p}_p'\right)$

$$E \cdot M = \frac{E'E}{c^2} - \underbrace{\vec{p}_e' \vec{p}_e}_{E'E \cos\vartheta} + \frac{E'}{\cancel{c}} \cdot M\cancel{c} - m_e^2 c^2$$

neglected

$$\Rightarrow \boxed{E' = \frac{E}{1 + \frac{E}{Mc^2}(1 - \cos\vartheta)}}$$

Recoil energy $= E - E'$

Fermi's "Golden Rule"

Matrix element (probability amplitude) $\mathcal{M}_{fi}$

$$\mathcal{M}_{fi} = \langle \psi_f | H_{int} | \psi_i \rangle = \int \psi_f^* H_{int} \psi_i \, dV$$

Reaction rate depends on the number of final states

Uncertainty relation => each particle occupies a volume $h^3 = (2\pi\hbar)^3$ in the phase space

Number of final states: $dn(p') = \frac{V}{(2\pi\hbar)^3} \cdot \underbrace{4\pi p'^2 dp'}_{\text{volume in momentum space}}$

$dE = v \, dp$

$$\varrho(E') = \frac{dn(E')}{dE'} = \frac{V \, 4\pi p'^2}{v' (2\pi\hbar)^3}$$

W - Reaction rate / target / projectile
— (transition probability)

$$\boxed{W = \frac{2\pi}{\hbar} |\mathcal{M}_{fi}|^2 \cdot \varrho(E')}$$ Fermi's Golden Rule

$$W = \frac{\dot{N}}{N_b \cdot N_a} = \frac{\dot{N} \cdot v_a}{N_b \frac{N_a}{V} \cdot n_a \cdot v_a} = \frac{\sigma \cdot v_a}{V}$$

$$W = \frac{\sigma \cdot v_a}{V} \Rightarrow$$

$$\boxed{\sigma = \frac{2\pi}{\hbar v_a} |\mathcal{M}_{fi}|^2 \cdot \varrho(E') \cdot V}$$

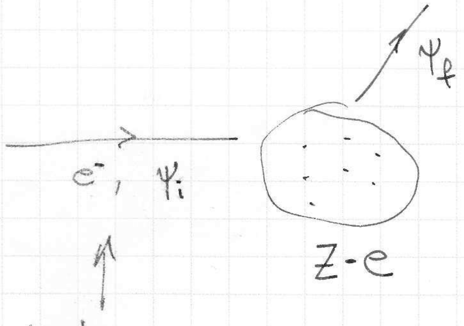
Measure σ and calculate $\mathcal{M}_{fi}$ or if H_{int} is known calculate σ

Rutherford X section

Scattering off an extended charge distribution

- Recoil negligible
- Born approximation $\qquad \psi_i = \frac{1}{\sqrt{V}} e^{i\bar{p}\bar{x}/\hbar} \; ; \; \psi_f = \frac{1}{\sqrt{V}} e^{i\bar{p}'\bar{x}/\hbar}$

$\bar{p}$ – 3 vector !



electron beam with n_a particles / V

$$\int_V |\psi_i|^2 \, dV = n_a \cdot V \qquad ; \qquad V = \frac{N_a}{n_a} \quad \text{(normalisation)}$$

$$\text{Fermi's GR} \Rightarrow \qquad \frac{\sigma v_a}{V} = W = \frac{2\pi}{\hbar} \left| \langle \psi_f | H_{int} | \psi_i \rangle \right|^2 \frac{dn}{dE_f}$$

E_f – Energy (total) of the final state

$$dn(|\vec{p}'|) = \frac{\underbrace{4\pi}_{d\Omega} |p'|^2 \, dp' \cdot V}{(2\pi\hbar)^3} \qquad \Rightarrow$$

$$d\sigma \cdot v_a \frac{1}{V} = \frac{2\pi}{\hbar} \left| \langle \psi_f | H_{int} | \psi_i \rangle \right|^2 \frac{V \cdot |p'|^2 \cdot dp'}{(2\pi\hbar)^3 \, dE_f} \, d\Omega$$

$$\frac{d\sigma}{d\Omega} = \frac{V^2 \cdot E'^2}{(2\pi)^2 (\hbar c)^4} \left| \langle \psi_f | H_{int} | \psi_i \rangle \right|^2 \qquad \left(E \simeq |\vec{p}| \cdot c \right)$$

$$v_a \simeq c$$

$H_{int} = e\phi \quad , \quad \phi$ - el. potential

$$\langle \psi_f | H_{int} | \psi_{in} \rangle = \frac{e}{V} \int e^{-i\vec{p}'\vec{x}/\hbar} \phi(x)\, e^{i\vec{p}\vec{x}/\hbar} d^3x = \circledast$$

Def: $\vec{q} = \vec{p} - \vec{p}'$

$$\circledast = \frac{e}{V} \int \phi(x)\, e^{i\vec{q}\vec{x}/\hbar} d^3x$$

Green's theorem : $\int (u \Delta v - v \Delta u) d^3x = 0$

$$u, v \xrightarrow[x \to \infty]{} 0$$

$$e^{i\vec{q}\vec{x}/\hbar} = -\frac{\hbar^2}{|\vec{q}|^2} \Delta e^{i\vec{q}\vec{x}/\hbar} \quad \Rightarrow$$

$$\langle \psi_f | H_{in} | \psi_i \rangle = -\frac{e\hbar^2}{V|\vec{q}|^2} \int \Delta\phi(x) \cdot e^{i\vec{q}\vec{x}/\hbar} d^3x$$

Poisson eq. $\Delta\phi = -\frac{\rho}{\varepsilon_0}$

$$\langle \psi_f | H_{in} | \psi_i \rangle = \frac{e\hbar^2}{\varepsilon_0 V \cdot |\vec{q}|^2} \int \rho(x)\, e^{i\vec{q}\vec{x}/\hbar} d^3x$$

Let: $\rho(x) = Z \cdot e \cdot f(x)$

$$= \frac{Z\, 4\pi\, \alpha \hbar^3 c}{V \cdot |\vec{q}|^2} \int f(x)\, e^{i\vec{q}\vec{x}/\hbar} d^3x$$

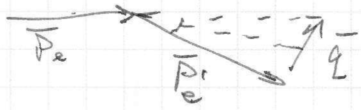
$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar}$$

$$F(\bar{q}) = \int f(x)\, e^{i\bar{q}\bar{x}/\hbar}\, d^3x$$ — Fourier transform of $f(x)$

↑

Form factor

$$\left(\frac{d\sigma}{d\Omega}\right)_R = \frac{4 Z^2 \alpha^2 (\hbar c)^2 E'^2}{|\bar{q}|^4 c^4}$$

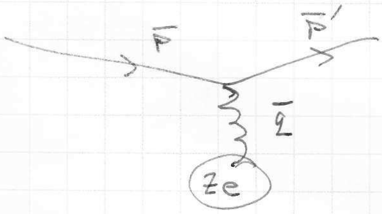


$|\bar{q}| = 2|\bar{p}| \sin\frac{\vartheta}{2}$

$|\bar{p}| \simeq \frac{E}{c}$

$$= \frac{Z^2 \alpha^2 (\hbar c)^2}{4 E^2 \sin^4\frac{\vartheta}{2}}$$

Remarques: de Brogle wave length $\lambda = \frac{\hbar}{|\bar{q}|} = \frac{\hbar}{|\bar{p}|\, 2 \sin\frac{\vartheta}{2}}$



q - small $\Rightarrow$ λ big — the structure of the nucleus (proton) can not be resolved

the nucleus is pointlike

=> Rutherford formula good approx

Mott X section (Rutheford + spin effects)

$$\left(\frac{d\sigma}{d\Omega}\right)_{Mott} = \left(\frac{d\sigma}{d\Omega}\right)_R \left(1 - \beta^2 \sin^2\frac{\vartheta}{2}\right) = \left(\frac{d\sigma}{d\Omega}\right)_{Ruth.} \cdot \cos^2\frac{\vartheta}{2}$$

if $\beta \simeq 1$

Back scattering ($\vartheta = 180°$) X section $= 0$!

Helicity – <u>conserved</u> quantity

$$h = \frac{\vec{S} \cdot \vec{P}}{|\vec{S}|\,|\vec{P}|}$$

Backscattering is only possible if the target has a spin as well.

<u>Form factors</u>

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott} \cdot |F(q^2)|^2$$

– Fourier transform of the charge distribution

Charge dist.:
- point like
- exponential
- Gauss
- homogeneous sphere

<u>Fig: 5.1 & 5.6</u>

Example ^{12}C measurement <u>Fig 5.5</u>

For a homogeneous sphere $\frac{|q| \cdot R}{\hbar} \sim 4.5$

The min for ^{12}C is at $\frac{q}{\hbar} \sim 1.8\ fm^{-1}$

$\Rightarrow R(^{12}C) = 4.5 \cdot \frac{\hbar}{|q|} = 2.5\ fm \quad (10^{-15}\ m)$

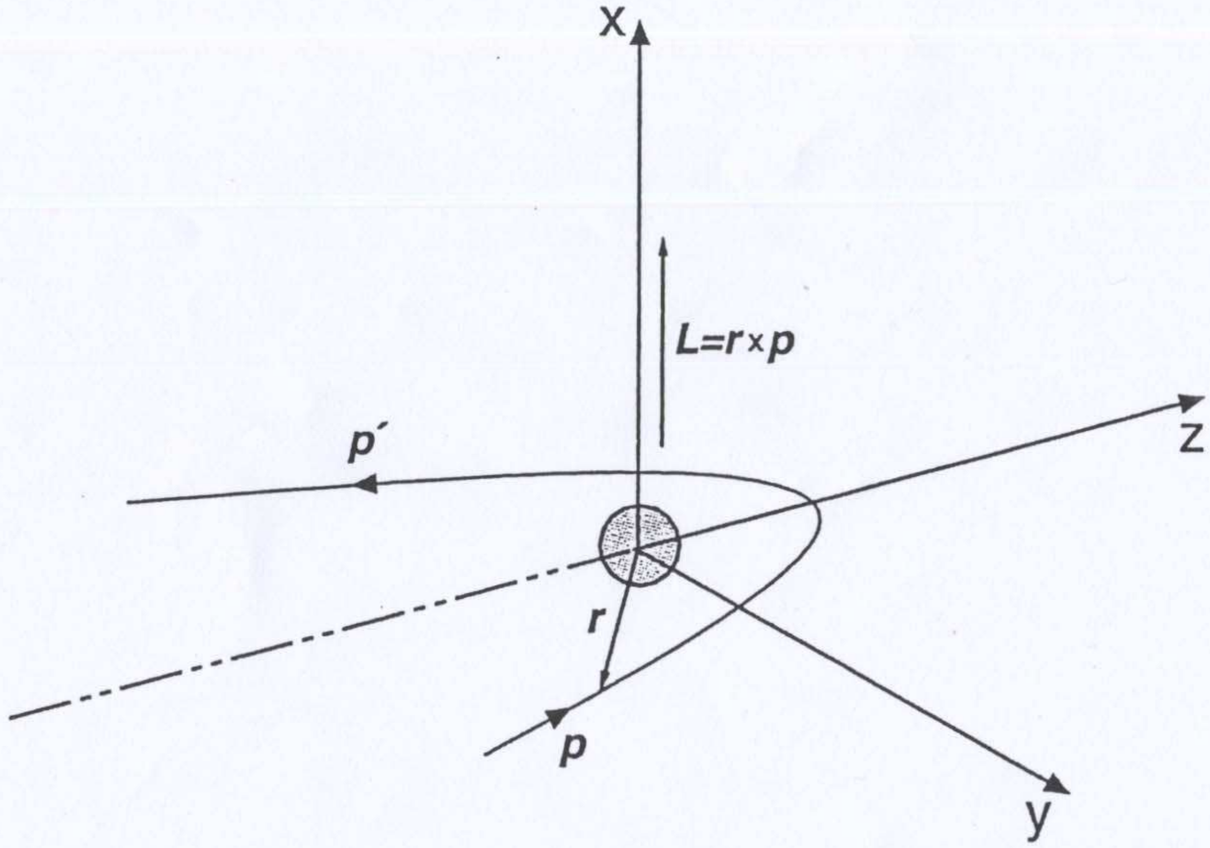


Fig. 5.3. Helicity, $h = \boldsymbol{s} \cdot \boldsymbol{p}/(|\boldsymbol{s}| \cdot |\boldsymbol{p}|)$, is conserved in the $\beta \to 1$ limit. This means that the spin projection on the z-axis would have to change its sign in scattering through 180°. This is impossible if the target is spinless, because of conservation of angular momentum.

Figure 5.3 shows the kinematics of scattering through 180°. We here

Table 5.1. Connection between charge distributions and form factors for some spherically symmetric charge distributions in Born approximation.

Charge distribution $f(r)$		Form Factor $F(q^2)$	
point	$\delta(r)/4\pi$	1	constant
exponential	$(a^3/8\pi) \cdot \exp(-ar)$	$\left(1 + q^2/a^2\hbar^2\right)^{-2}$	dipole
Gaussian	$\left(a^2/2\pi\right)^{3/2} \cdot \exp\left(-a^2r^2/2\right)$	$\exp\left(-q^2/2a^2\hbar^2\right)$	Gaussian
homogeneous sphere	$\begin{cases} 3/4\pi R^3 & \text{for } r \leq R \\ 0 & \text{for } r > R \end{cases}$	$3\,\alpha^{-3}\,(\sin\alpha - \alpha\cos\alpha)$ with $\alpha = \lvert q\rvert R/\hbar$	oscillating

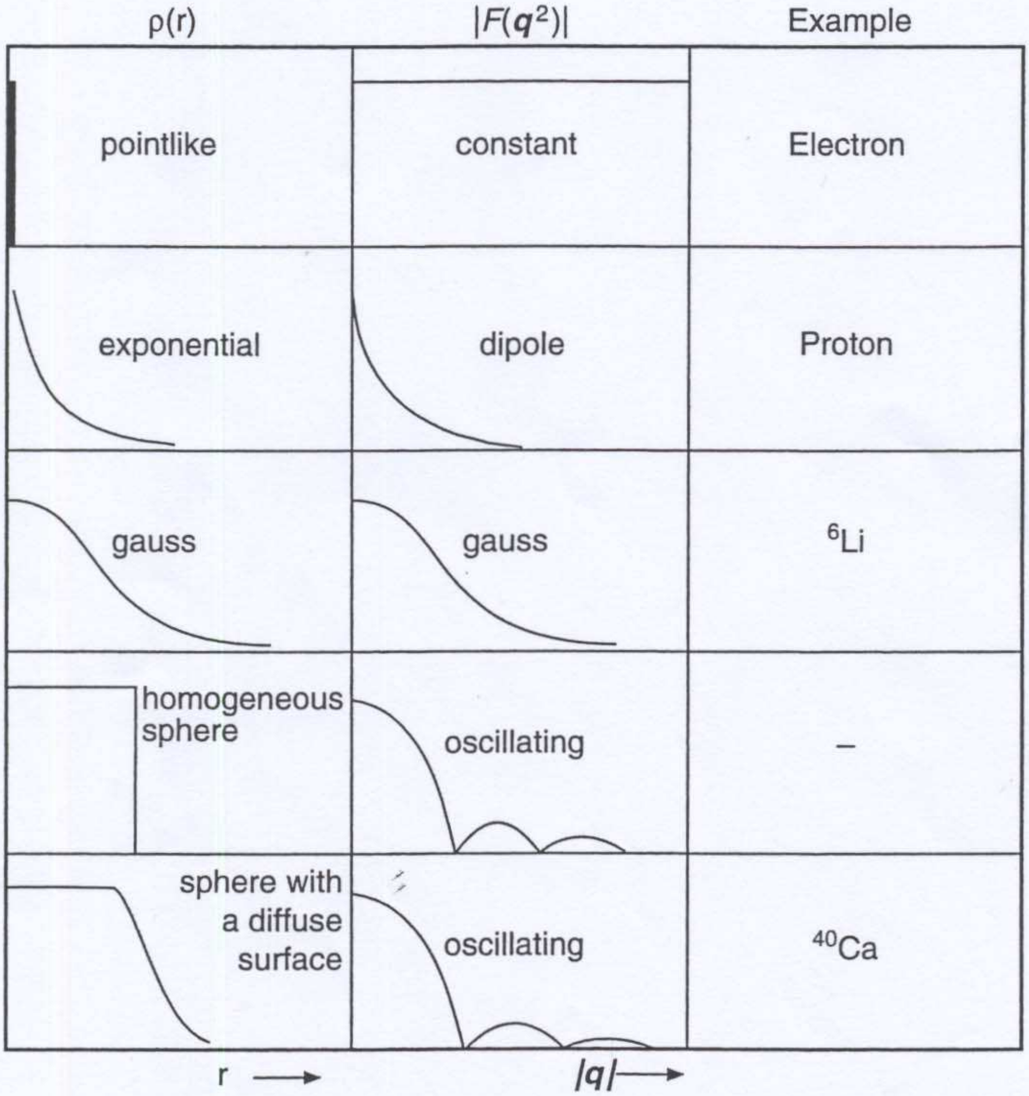


Fig. 5.6. Relation between the radial charge distribution $\varrho(r)$ and the corresponding form factor in Born approximation. A constant form factor corresponds to a pointlike charge (e. g., an electron); a dipole form factor to a charge distribution which falls off exponentially (e. g., a proton); a Gaussian form factor to a Gaussian charge distribution (e. g., ^{6}Li nucleus); and an oscillating form factor corresponds to a homogeneous sphere with a more or less sharp edge. All nuclei except for the lightest ones, display an oscillating form factor.

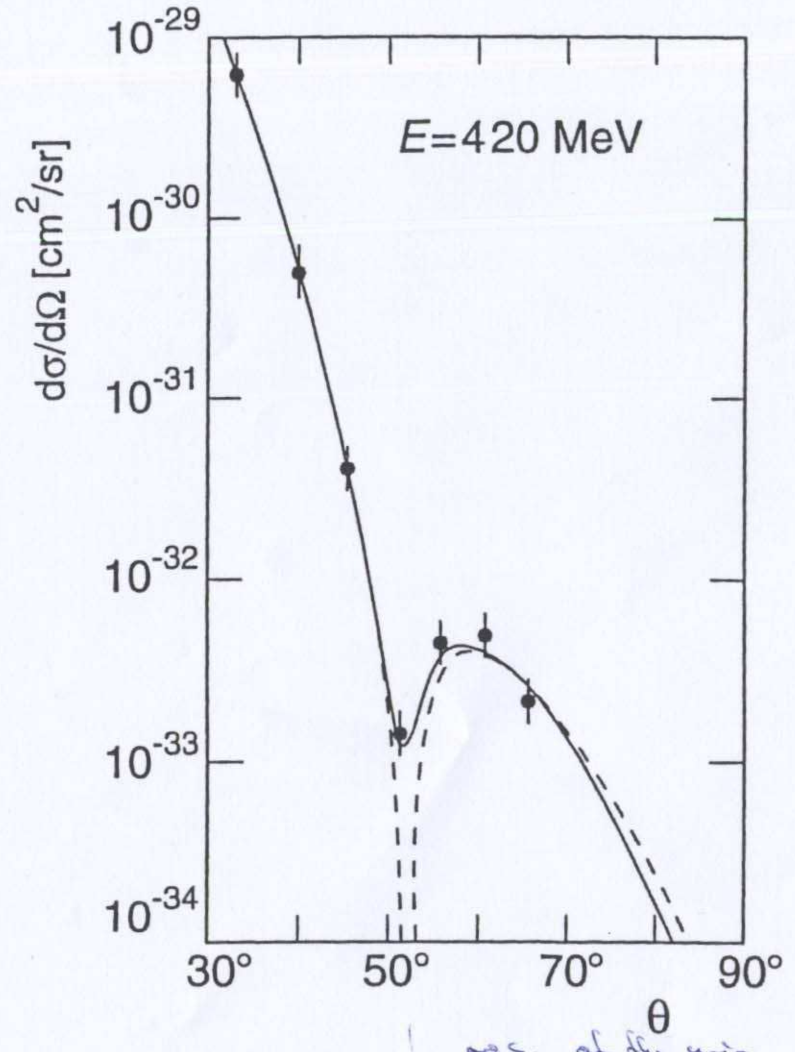


Fig. 5.5. Measurement of the form factor of ^{12}C by electron scattering (from [Ho57]). The figure shows the differential cross-section measured at a fixed beam energy of 420 MeV, at 7 different scattering angles. The dashed line corresponds to scattering of a plane wave off an homogeneous sphere with a diffuse surface (Born approximation). The solid line corresponds to an exact phase shift analysis which was fitted to the experimental data.

pos. of the min => $R(^{12}C) = 2.5$ fm

the theoretical prediction for $F(\boldsymbol{q}^2)$ and varies the parameters to obtain a best fit between theory and the measured value of $F(\boldsymbol{q}^2)$.

The form factor can be calculated analytically for certain charge distributions described by some simple radial functions